確率統計学概論　解答例

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■　I_n （$n \in \mathbb{N}$）を

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

により定めるとき，I_n を n を用いて表せ．また，I_n をガンマ関数 $\Gamma(z)$ を用いて表せ．

（解） $\Gamma(z+1) = z\,\Gamma(z)$ （$z>0$）より，

$$\begin{aligned}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} =& \frac{\frac{n-1}{2}\cdot\Gamma(\frac{n-1}{2})}{\frac{n-2}{2}\cdot\Gamma(\frac{n-2}{2})} = \frac{n-1}{n-2}\cdot\frac{\frac{n-3}{2}\cdot\Gamma(\frac{n-3}{2})}{\frac{n-4}{2}\cdot\Gamma(\frac{n-4}{2})} = \frac{n-1}{n-2}\cdot\frac{n-3}{n-4}\cdot\frac{\frac{n-5}{2}\cdot\Gamma(\frac{n-5}{2})}{\frac{n-6}{2}\cdot\Gamma(\frac{n-6}{2})} = \cdots \\ =& \frac{n-1}{n-2}\cdot\frac{n-3}{n-4}\cdot\frac{n-5}{n-6}\cdots\begin{cases}\dfrac{4}{3}\cdot\dfrac{2}{1}\cdot\dfrac{\Gamma(\frac{2}{2})}{\Gamma(\frac{1}{2})} & (n \text{ が奇数のとき}) \\ \dfrac{5}{4}\cdot\dfrac{3}{2}\cdot\dfrac{\frac{1}{2}\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{2})} & (n \text{ が偶数のとき})\end{cases}\end{aligned}$$

が成り立つ．I_1 および I_2 はそれぞれ

$$I_1 = \int_0^{\pi/2} [-\cos x]_0^{\pi/2} = 1, \qquad I_2 = \int_0^{\pi/2} \frac{1-\cos 2x}{2}\,dx = \left[\frac{2x - \sin 2x}{4}\right]_0^{\pi/2} = \frac{\pi}{4}$$

である．また，各 $n \in \mathbb{N}$ に対して，部分積分法により

$$\begin{aligned}I_{n+2} =& \int_0^{\pi/2} [-\cos x]' \cdot \sin^{n+1} x\,dx = (n+1)\int_0^{\pi/2} \sin^n x\,\cos^2 x\,dx \\ =& (n+1)\int_0^{\pi/2} \sin^n x\,(1-\sin^2 x)\,dx = (n+1)\,I_n - (n+1)\,I_{n+2}\end{aligned}$$

となるので，I_n は漸化式

$$I_1 = 1, \qquad I_2 = \frac{\pi}{4}, \qquad I_{n+2} = \frac{n+1}{n+2}\cdot I_n \qquad (n \in \mathbb{N})$$

をみたす．$\Gamma(1/2) = \sqrt{\pi}$，$\Gamma(1) = 1$ より，n が奇数のとき

$$\begin{aligned}I_n =& \frac{n-1}{n}\,I_{n-2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot I_{n-4} = \cdots \\ =& \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot I_1 = \frac{1}{n}\cdot\frac{n-1}{n-2}\cdot\frac{n-3}{n-4}\cdots\frac{4}{3}\cdot\frac{2}{1} = \frac{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}{n\,\Gamma(\frac{n}{2})}\end{aligned}$$

であり，n が偶数のとき

$$\begin{aligned}I_n =& \frac{n-1}{n}\,I_{n-2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot I_{n-4} = \cdots \\ =& \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot I_2 = \frac{1}{n}\cdot\frac{n-1}{n-2}\cdot\frac{n-3}{n-4}\cdots\frac{5}{4}\cdot\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}\cdot\sqrt{\pi} = \frac{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}{n\,\Gamma(\frac{n}{2})}\end{aligned}$$

である．■