応用数学 I　解答例

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■　I_n （$n \in \mathbb{N}$）を $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ により定義するとき，

$$I_n = \frac{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}{n\,\Gamma(\frac{n}{2})}$$

と表されることを示せ．

（解）　部分積分法により

$$\begin{aligned}
I_{n+2} &= \int_0^{\frac{\pi}{2}} \sin^{n+1} x\ [-\cos x]'\ dx \\
&= \left[\sin^{n+1} x\ (-\cos x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n+1)\ \sin^n x\ \cos\ x\ (-\cos x)\ dx \\
&= (n+1)\ \int_0^{\frac{\pi}{2}} \sin^n x\ \cos^2\ x\,dx = (n+1)\ I_n - (n+1)\ I_{n+2}
\end{aligned}$$

となるので，

$$I_{n+2} = \frac{n+1}{n+2}\,I_n, \qquad n \in \mathbb{N}$$

が得られる．また，

$$\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = 1, \\
I_2 &= \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\,x}{2}\,dx = \left[\frac{x}{2} - \frac{\sin 2\,x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}
\end{aligned}$$

である．$\Gamma(\frac{1}{2}) = \sqrt{\pi}$，$\Gamma(1) = 1$ であり，すべての $z > 0$ に対して $\Gamma(z+1) = z\,\Gamma(z)$ が成り立つので，n が奇数のとき

$$\begin{aligned}
I_n =& \frac{n-1}{n}\,I_{n-2} = \cdots = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 \\
=& \frac{\sqrt{\pi}}{n}\cdot\frac{\left(\frac{n+1}{2}-1\right)\cdot\left(\frac{n+1}{2}-2\right)\cdots 2\cdot 1}{\left(\frac{n}{2}-1\right)\cdot\left(\frac{n}{2}-2\right)\cdots\frac{1}{2}\cdot\sqrt{\pi}} = \frac{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}{n\,\Gamma(\frac{n}{2})}
\end{aligned}$$

となり，n が偶数のとき

$$\begin{aligned}
I_n =& \frac{n-1}{n}\,I_{n-2} = \cdots = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{\pi}{4} \\
=& \frac{\sqrt{\pi}}{n}\cdot\frac{\left(\frac{n+1}{2}-1\right)\cdot\left(\frac{n+1}{2}-2\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{\left(\frac{n}{2}-1\right)\cdot\left(\frac{n}{2}-2\right)\cdots 1} = \frac{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})}{n\,\Gamma(\frac{n}{2})}
\end{aligned}$$

となる．■