応用数学 I　解答例

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問題　自然数 n に対して

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

とおくとき，

$$I_{n+2} = \frac{n+1}{n+2} I_n, \qquad I_n = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{n\,\Gamma\left(\frac{n}{2}\right)}$$

と表されることを示せ．ここで，$\Gamma(x)$ はガンマ関数であり，必要があれば $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma(1) = 1$, $\Gamma(x+1) = x\,\Gamma(x)$ $(\forall x > 0)$ を用いてもよい．

（解） 部分積分法により

$$\begin{aligned}
I_{n+1} &= \int_0^{\frac{\pi}{2}} \sin x \, \sin^{n+1} x \, dx = \int_0^{\frac{\pi}{2}} [-\cos x]' \sin^{n+1} x \, dx \\
&= \big[(-\cos x)\,\sin^{n+1} x\big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x)\,\{(n+1)\,\sin^n x\,\cos x\}\,dx \\
&= (n+1)\int_0^{\frac{\pi}{2}} \sin^n x\,\left(1 - \sin^2 x\right)\,dx = (n+1)\,I_n - (n+1)\,I_{n+2}
\end{aligned}$$

となり，$I_{n+2} = \dfrac{n+1}{n+2} I_n$ が得られる．ガンマ関数 $\Gamma(x)$ がみたす関係式より，すべての自然数 n に対して

$$\Gamma(n+1) = n!, \qquad \Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\cdot\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

が成り立つ．

$$\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = 1, \\
I_2 &= \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}
\end{aligned}$$

より，$n = 2\ell - 1$ $(\ell \in \mathbb{N})$ のとき

$$\begin{aligned}
I_n &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\cdots\cdot\frac{2}{3} = \frac{\ell-1}{\ell-\frac{1}{2}}\cdot\frac{\ell-2}{(\ell-1)-\frac{1}{2}}\cdot\frac{\ell-3}{(\ell-1)-\frac{3}{2}}\cdot\cdots\cdot\frac{1}{\frac{3}{2}} \\
&= \frac{\Gamma(\ell)}{\left(\ell-\frac{1}{2}\right)\,\Gamma\left((\ell-1)+\frac{1}{2}\right)\,\frac{2}{\sqrt{\pi}}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{n\,\Gamma\left(\frac{n}{2}\right)}
\end{aligned}$$

となり，$n = 2\ell$ $(\ell \in \mathbb{N})$ のとき

$$\begin{aligned}
I_n &= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\cdots\cdot\frac{3}{4}\cdot\frac{\pi}{4} = \frac{\ell-\frac{1}{2}}{\ell}\cdot\frac{\ell-\frac{3}{2}}{\ell-1}\cdot\frac{\ell-\frac{5}{2}}{\ell-2}\cdot\cdots\cdot\frac{\frac{3}{2}}{2}\cdot\frac{\pi}{4} \\
&= \frac{\Gamma(\ell+\frac{1}{2})\,\frac{2}{\sqrt{\pi}}}{\ell\,\Gamma(\ell)}\cdot\frac{\pi}{4} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{n\,\Gamma\left(\frac{n}{2}\right)}
\end{aligned}$$

となる．■