応用数学 I　解答例

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問題　確率変数 X_1，X_2，$\cdots$，X_n は互いに独立で，平均 μ，分散 σ^2 の同じ確率分布に従うとき，

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k, \qquad U^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(X_k - \bar{X}\right)^2$$

の期待値を求めよ．

(解) 期待値の線形性より

$$E\left[\bar{X}\right] = E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \frac{1}{n}\sum_{k=1}^{n} E\left[X_k\right] = \mu$$

である．互いに独立であるから，$k \neq \ell$ のとき

$$E\left[(X_k-\mu)\,(X_\ell-\mu)\right] = 0$$

であることに注意したい．

$$\bar{X} - \mu = \frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)$$

より，

$$\begin{aligned}
E\left[(X_k-\mu)\,\left(\bar{X}-\mu\right)\right] &= E\left[(X_k-\mu)\left\{\frac{1}{n}\sum_{\ell=1}^{n}(X_\ell-\mu)\right\}\right] = \frac{1}{n}\sum_{\ell=1}^{n}E\left[(X_k-\mu)\,(X_\ell-\mu)\right] \\
&= \frac{1}{n}\left(E\left[(X_k-\mu)^2\right] + \sum_{\ell\neq k}E\left[(X_k-\mu)\,(X_\ell-\mu)\right]\right) = \frac{\sigma^2}{n}, \\
E\left[\left(\bar{X}-\mu\right)^2\right] &= E\left[\left\{\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)\right\}\left\{\frac{1}{n}\sum_{\ell=1}^{n}(X_\ell-\mu)\right\}\right] \\
&= \frac{1}{n^2}\sum_{k=1}^{n}\sum_{\ell=1}^{n}E\left[(X_k-\mu)\,(X_\ell-\mu)\right] \\
&= \frac{1}{n^2}\left\{\sum_{k=1}^{n}E\left[(X_k-\mu)^2\right] + \sum_{\ell\neq k}E\left[(X_k-\mu)\,(X_\ell-\mu)\right]\right\} = \frac{\sigma^2}{n}
\end{aligned}$$

が得られ，

$$\begin{aligned}
E\left[U^2\right] &= \frac{1}{n-1}\sum_{k=1}^{n}E\left[\left(X_k-\bar{X}\right)^2\right] = \frac{1}{n-1}\sum_{k=1}^{n}E\left[\left\{(X_k-\mu)-\left(\bar{X}-\mu\right)\right\}^2\right] \\
&= \frac{1}{n-1}\sum_{k=1}^{n}\left\{E\left[(X_k-\mu)^2\right] - 2\,E\left[(X_k-\mu)\,\left(\bar{X}-\mu\right)\right] + E\left[\left(\bar{X}-\mu\right)^2\right]\right\} \\
&= \frac{1}{n-1}\sum_{k=1}^{n}\left(\sigma^2 - 2\cdot\frac{\sigma^2}{n} + \frac{\sigma^2}{n}\right) = \frac{1}{n-1}\cdot\left(\sigma^2 - \frac{\sigma^2}{n}\right)\cdot n = \sigma^2
\end{aligned}$$

となる．■