確率統計学１　解答例

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■　関係式

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n t\, dt = \frac{\sqrt{\pi}\,\Gamma((n+1)/2)}{n\,\Gamma(n/2)}, \qquad n = 1, 2, 3, \cdots$$

が成り立つことを示せ．

（解）$n \in \mathbb{N}$ を $m \in \mathbb{Z}$，$k \in \{2, 3\}$ を用いて $n = 2\,m + k$ と表すことができる．

$$I_{k-2} = \begin{cases} \pi/2 & (k = 2) \\ 1 & (k = 3) \end{cases}$$

$$\frac{2\,\Gamma(k/2)}{\Gamma((k-1)/2)} = \begin{cases} \dfrac{2\,\Gamma(1)}{\Gamma(1/2)} = \dfrac{2}{\sqrt{\pi}} & (k = 2) \\ \dfrac{2\,\Gamma(3/2)}{\Gamma(1)} = \Gamma(1/2) = \sqrt{\pi} & (k = 3) \end{cases}$$

より

$$\frac{2\,\Gamma(k/2)}{\Gamma((k-1)/2)} \cdot I_{k-2} = \sqrt{\pi}$$

である．$k/2 = n/2 - m$ より

$$\begin{aligned} I_n = \int_0^{\frac{\pi}{2}} \sin^n t\, dt &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{k-1}{k} \cdot I_{k-2} \\ &= \frac{(n-1)/2}{n/2} \cdot \frac{(n-1)/2 - 1}{n/2 - 1} \cdot \frac{(n-1)/2 - 2}{n/2 - 2} \cdots \frac{(n-1)/2 - m}{n/2 - m} \cdot \frac{\sqrt{\pi}\,\Gamma((n-1)/2 - m)}{2\,\Gamma(n/2 - m)} \\ &= \frac{\sqrt{\pi}\,\Gamma((n-1)/2 + 1)}{2\,\Gamma(n/2 + 1)} = \frac{\sqrt{\pi}\,\Gamma((n+1)/2)}{n\,\Gamma(n/2)} \end{aligned}$$

となる．■