解析学１　解答例

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■　関数 $f(x,y)$ の 2 回までのすべての偏導関数が存在し，それらは連続であるとする．このとき，$g(r,\theta) = f(r\,\cos\theta, r\,\sin\theta)$ の偏導関数 $\dfrac{\partial g}{\partial r}(r,\theta)$，$\dfrac{\partial g}{\partial \theta}(r,\theta)$ および 2 階偏導関数 $\dfrac{\partial^2 g}{\partial r^2}(r,\theta)$ を求めよ．

（解）　合成関数の微分法により

$$
\begin{aligned}
\frac{\partial g}{\partial r}(r,\theta) =& \frac{\partial f}{\partial x}(r\,\cos\theta, r\,\sin\theta)\,\cos\theta + \frac{\partial f}{\partial y}(r\,\cos\theta, r\,\sin\theta)\,\sin\theta \\
\frac{\partial g}{\partial \theta}(r,\theta) =& \frac{\partial f}{\partial x}(r\,\cos\theta, r\,\sin\theta)\,(-r\,\cos\theta) + \frac{\partial f}{\partial y}(r\,\cos\theta, r\,\sin\theta)\,r\,\sin\theta \\
=& -\frac{\partial f}{\partial x}(r\,\cos\theta, r\,\sin\theta)\,r\,\sin\theta + \frac{\partial f}{\partial y}(r\,\cos\theta, r\,\sin\theta)\,r\,\cos\theta, \\
\frac{\partial^2 g}{\partial r^2}(r,\theta) =& \left\{ \frac{\partial^2 f}{\partial x^2}(r\,\cos\theta, r\,\sin\theta)\,\cos\theta + \frac{\partial^2 f}{\partial y \partial x}(r\,\cos\theta, r\,\sin\theta)\,\sin\theta \right\} \cos\theta \\
&+ \left\{ \frac{\partial^2 f}{\partial x \partial y}(r\,\cos\theta, r\,\sin\theta)\,\cos\theta + \frac{\partial^2 f}{\partial y^2}(r\,\cos\theta, r\,\sin\theta)\,\sin\theta \right\} \sin\theta \\
=& \frac{\partial^2 f}{\partial x^2}(r\,\cos\theta, r\,\sin\theta)\,\cos^2\theta \\
&+ 2\,\frac{\partial^2 f}{\partial x \partial y}(r\,\cos\theta, r\,\sin\theta)\,\sin\theta\,\cos\theta + \frac{\partial^2 f}{\partial y^2}(r\,\cos\theta, r\,\sin\theta)\,\sin^2\theta
\end{aligned}
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