解析学１　解答例

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■　関数

$$f(x,y) = \begin{cases} \dfrac{x\,y\,(x^2-y^2)}{x^2+y^2} & ((x,y) \neq (0,0)) \\ 0 & ((x,y) = (0,0)) \end{cases}$$

に対して $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ と $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ は一致するか否か調べよ．

（解）　$(x,y) \neq (0,0)$ において偏導関数を求めると，

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) =& \frac{\partial}{\partial x}\left[\frac{x\,y\,(x^2-y^2)}{x^2+y^2}\right] = \frac{y\,(x^4+4\,x^2\,y^2-y^4)}{(x^2+y^2)^2}, \\ \frac{\partial f}{\partial y}(x,y) =& \frac{\partial}{\partial y}\left[\frac{x\,y\,(x^2-y^2)}{x^2+y^2}\right] = \frac{x\,(x^4-4\,x^2\,y^2-y^4)}{(x^2+y^2)^2} \end{aligned}$$

となる．$(x,y) = (0,0)$ において偏微分係数を求めると，

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0, \\ \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0 \end{aligned}$$

であるから，

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y}(0,0) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{h^5}{h^4} - 0\right) = \lim_{h\to 0} 1 = 1, \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial y}(0,0)\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{-h^5}{h^4} - 0\right) = \lim_{h\to 0}(-1) = -1 \end{aligned}$$

となり，$\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ と $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ は異なる．■