解析学 1　解答例

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■ 関数 $f(x,y)$ を

$$f(x,y)=\begin{cases}\dfrac{x\,y\,(x^2-y^2)}{x^2+y^2} & ((x,y)\neq(0,0))\\[2ex] 0 & ((x,y)=(0,0))\end{cases}$$

により定義するとき，$\dfrac{\partial f}{\partial x}(0,0)$，$\dfrac{\partial f}{\partial y}(0,0)$，$\dfrac{\partial^2 f}{\partial x\partial y}(0,0)$，$\dfrac{\partial^2 f}{\partial y\partial x}(0,0)$ を求めよ．

（解） $(x,y)\neq(0,0)$ のとき

$$\frac{\partial f}{\partial x}(x,y)=\frac{y\,(x^4+4\,x^2\,y^2-y^4)}{(x^2+y^2)^2},\qquad \frac{\partial f}{\partial y}(x,y)=\frac{x\,(x^4-4\,x^2\,y^2-y^4)}{(x^2+y^2)^2}$$

であるから，

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0)&=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{0-0}{h}=0,\\
\frac{\partial f}{\partial y}(0,0)&=\lim_{h\to0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to0}\frac{0-0}{h}=0,\\
\frac{\partial^2 f}{\partial x\partial y}(0,0)&=\lim_{h\to0}\frac{1}{h}\left\{\frac{\partial f}{\partial y}(h,0)-\frac{\partial f}{\partial y}(0,0)\right\}=\lim_{h\to0}\frac{1}{h}\,(h-0)=1,\\
\frac{\partial^2 f}{\partial y\partial x}(0,0)&=\lim_{h\to0}\frac{1}{h}\left\{\frac{\partial f}{\partial x}(0,h)-\frac{\partial f}{\partial x}(0,0)\right\}=\lim_{h\to0}\frac{1}{h}\,\{(-h)-0\}=-1
\end{aligned}$$

となる． ■