解析学１　解答例

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■　2 変数関数 $h(x,y)$ を

$$h(x,y) = \begin{cases} \dfrac{x\,y^2}{x^2+y^2} & ((x,y) \neq (0,0) \text{ のとき}) \\ 0 & ((x,y) = (0,0) \text{ のとき}) \end{cases}$$

で定めるとき，

$$\frac{\partial h}{\partial x}(0,0), \qquad \frac{\partial h}{\partial y}(0,0), \qquad \frac{\partial^2 h}{\partial x^2}(0,0), \qquad \frac{\partial^2 h}{\partial x \partial y}(0,0), \qquad \frac{\partial^2 h}{\partial y \partial x}(0,0), \qquad \frac{\partial^2 h}{\partial y^2}(0,0)$$

が存在するかどうか調べよ．

（解）　商の微分公式より，$(x,y) \neq (0,0)$ のとき

$$\begin{aligned} \frac{\partial h}{\partial x}(x,y) &= \frac{y^2 \cdot (x^2+y^2) - (x\,y^2) \cdot (2\,x)}{(x^2+y^2)^2} = \frac{y^2\,(y^2-x^2)}{(x^2+y^2)^2}, \\ \frac{\partial h}{\partial y}(x,y) &= \frac{(2\,x\,y) \cdot (x^2+y^2) - (x\,y^2) \cdot (2\,y)}{(x^2+y^2)^2} = \frac{2\,x^3\,y}{(x^2+y^2)^2} \end{aligned}$$

であることに注意すると，$t \neq 0$ のとき

$$h(t,0) = 0, \qquad h(0,t) = 0, \qquad \frac{\partial h}{\partial x}(t,0) = 0, \qquad \frac{\partial h}{\partial x}(0,t) = 1, \qquad \frac{\partial h}{\partial y}(t,0) = 0, \qquad \frac{\partial h}{\partial y}(0,t) = 0$$

であるから，

$$\begin{aligned} \frac{\partial h}{\partial x}(0,0) &= \lim_{t\to0} \frac{h(t,0)-h(0,0)}{t} = \lim_{t\to0} \frac{0-0}{t} = 0, \\ \frac{\partial h}{\partial y}(0,0) &= \lim_{t\to0} \frac{h(0,t)-h(0,0)}{t} = \lim_{t\to0} \frac{0-0}{t} = 0, \\ \frac{\partial^2 h}{\partial x^2}(0,0) &= \lim_{t\to0} \left[\frac{1}{t} \left\{ \frac{\partial h}{\partial x}(t,0) - \frac{\partial h}{\partial x}(0,0) \right\} \right] = \lim_{t\to0} \frac{0-0}{t} = 0, \\ \frac{\partial^2 h}{\partial x \partial y}(0,0) &= \lim_{t\to0} \left[\frac{1}{t} \left\{ \frac{\partial h}{\partial y}(t,0) - \frac{\partial h}{\partial y}(0,0) \right\} \right] = \lim_{t\to0} \frac{0-0}{t} = 0, \\ \frac{\partial^2 h}{\partial y^2}(0,0) &= \lim_{t\to0} \left[\frac{1}{t} \left\{ \frac{\partial h}{\partial y}(0,t) - \frac{\partial h}{\partial y}(0,0) \right\} \right] = \lim_{t\to0} \frac{0-0}{t} = 0 \end{aligned}$$

であるが，

$$\frac{\partial^2 h}{\partial y \partial x}(0,0) = \lim_{t\to0} \left[\frac{1}{t} \left\{ \frac{\partial h}{\partial x}(0,t) - \frac{\partial h}{\partial x}(0,0) \right\} \right] = \lim_{t\to0} \frac{1-0}{t} = \lim_{t\to0} \frac{1}{t}$$

より $\dfrac{\partial^2 h}{\partial y \partial x}(0,0)$ は存在しない．■